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Hypersonic liquid jets

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The problem of a liquid jet moving at hypersonic speed into a gas is considered in a frame of reference in which the tip of the jet is at rest. The liquid jet flow is assumed to be inviscid, irrotational, incompressible and two-dimensional although an approximate extension to the axially symmetric case is developed. The air flow in the hypersonic shock layer is analysed using the modified Newtonian theory. The condition of continuity of pressure at the gas–liquid interface then allows a solution to the potential problem in the liquid to be found by transforming to the hodograph plane. The resulting jet shape is presented graphically in terms of the relevant parameters.

The application of the method to penetration problems is also discussed and comparisons made with experimental results and 'exact' solutions.

1. Introduction

In recent years there has been an increasing use of high-speed water jets in the cutting of coal, particularly in the U.S.S.R., U.S.A. and Poland. This method of cutting the rock is free from the hazard of methane ignition and consequently is a subject in which there is a great deal of interest.

For soft rocks, such as coal, with low compressive strengths, quite large jets at a pressure below about 300 atmospheres were sufficient to achieve the cutting. Such jets issued into the atmosphere with a Mach number, based on the ratio of jet speed to ambient-gas sound speed, of less than unity. However, much higher pressures would be required for the cutting of harder rocks, resulting in free-stream Mach numbers for the jets of 3 or more. Leach & Walker (1966) carried out experiments with such jets and examined their penetration into several types of rock.

This physical situation, apart from the rock penetration, is typified by a liquid jet moving at hypersonic speed into a gas at rest and such a model will be analysed in this paper in a frame of reference in which the tip of the jet is stationary.

A similar situation arises in a type of re-entry problem examined experimentally by Finlay (1966), in which a jet of gas was emitted from the blunt nose of a re-entry vehicle to reduce the heat transfer from the shock layer to the body. In this case the speed of the jet was such that reattachment to the body took place in a region where the pressure had dropped quite markedly from its stagnation value. It was in this context that the interaction of an incompressible jet opposing a hypersonic stream was analysed by Lam (1959)† and the model developed was almost identical to the approach used in this paper.

The penetration of solids by high-speed liquid jets, or solids, also has certain simi-

† The author is indebted to Professor N. C. Freeman for this observation.

larities with the problem just outlined. Pack & Evans (1951) established the notion of a hydrodynamic analogy as an acceptable simplification for such phenomena when the dynamic pressure generated by the jet greatly exceeded the yield strength of the material. They were concerned with the penetration of metal by explosively generated liquid-metal Munroe jets. This work has been extended by Tate (1967, 1969) to examine the deceleration of long metal rods after striking a target. The theoretical aspects of these studies were limited to one-dimensional considerations. Extensive numerical calculations of the flow producing the Munroe jet and the penetration of the jet into a target have been carried out by Harlow & Pracht (1966) using the particle-in-cell method. Their comprehensive results show the shape of the penetrating jet to be of the same general form as that produced in Finlay's experiments. That is to say, the jet is 'mushroom shaped' near the nose, presumably owing to the dominance in these problems of the dynamic rather than the viscous effects.

The primary problem considered in the present paper is that of a two-dimensional liquid jet in a hypersonic stream of gas. The relevance of this approach to more general situations will be discussed. For example, a simple momentum-integral approach allows certain features of the axially symmetric jet to be determined with some confidence. It is also argued that the results obtained for the hypersonic jet are of value in forming the basis of a solution to the penetration problem.

For simplicity it will be assumed that the liquid comprising the jet is incompressible, inviscid and irrotational. As there are no rigid boundaries in the problem, the main effect of viscosity will be confined to the mixing region between the liquid in the jet and the gas through which the jet is travelling. Across the mixing layer the pressure is approximately constant, so that, provided that a condition of continuity of pressure across the gas–liquid interface is enforced, no serious errors should arise from the neglect of viscosity and the free shear layer need not be considered in detail.

The flow is taken to be steady and the velocity profile across the jet at large distances from its nose is uniform. In such a jet the flow will therefore remain irrotational.

In the hypersonic flow field, use is made of the 'modified' Newtonian theory. The value of this approach for determining the pressure distribution on the body surface is well recognized, although it must be accepted that the so-called theory is empirical.

2. Method of solution

The type of flow under consideration is shown diagrammatically in figure 1. The mushroom-shaped region occupied by the liquid jet is of the form determined experimentally by Finlay, and as has been pointed out, it is also of the same general shape as that indicated by the numerical results of Harlow & Pracht for the liquid-jet penetration calculation.

The present problem essentially reduces to the determination of the free surfaces which separate the various regions. The bow shock S (figure 1) separates the uniform stream from the hypersonic shock layer, which, in turn, is bounded by the gas–liquid interface, which effectively acts as a body. The other free boundary of the liquid jet is assumed to be adjacent to a region of uniform pressure.

First of all, considering the hypersonic shock layer, the only analytical method for this region which is uniformly valid in some limit is that typified by the analysis of Freeman (1956). A considerable simplification which results is the fact that the

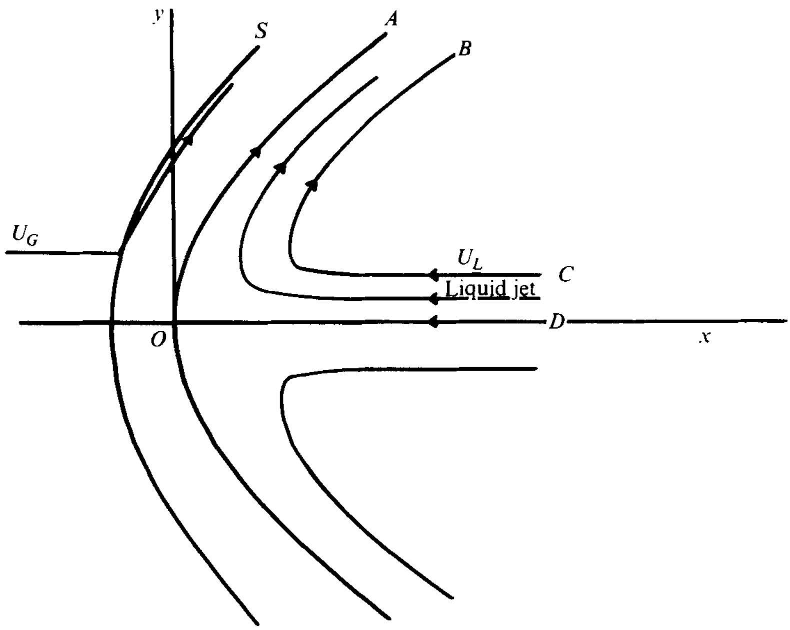

Figure 1. *Diagrammatic representation of the flow* field showing the co-ordinate axes.

shock and body, the gas–liquid interface in this instance, have the same shape to first order. However, there is a difficulty associated with the first approximation in that it predicts an unrealistic fall to zero of the surface pressure on most bluff bodies. Clearly the flow in the liquid jet is essentially determined by this pressure distribution. Hence, in the present analysis, it seems more appropriate to use the 'modified' Newtonian pressure distribution proposed by Lees (1955), which shows excellent agreement both with experimental results and with 'exact' calculations for hypersonic flow past bluff convex bodies. The reasons for its success are well set out by Hayes & Probstein (1966, p. 401).

Consequently, if Φ is the inclination of the body surface to the mainstream, the surface pressure distribution is taken to be

$$P/\rho_G U_G^2 = \sin^2\Phi, \tag{1}$$

where ρ and U denote density and free-stream velocity respectively and the suffix G refers to the gas.

In the body of the liquid jet the flow is assumed to be inviscid, incompressible and irrotational, so that

$$\nabla^2\psi_L = 0, \tag{2}$$

where ψ_L is a stream function for the liquid and is defined by

$$q\cos\theta = \partial\psi_L/\partial y, \quad q\sin\theta = -\partial\psi_L/\partial x, \tag{3}$$

where x and y are as shown in figure 1, q is the speed of the liquid and θ the inclination of the velocity to the x axis.

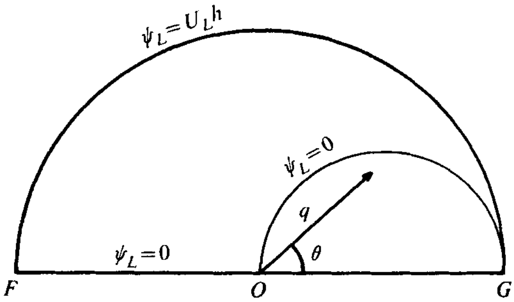

FIGURE 2. The jet flow field in the hodograph plane.

At the jet surface there is no component of velocity normal to the surface and the tangential component is determined by Bernoulli's equation applied along the streamlines forming the surface.

Along the surface denoted in figure 1 by OA, the speed q is given by

$$\tfrac{1}{2}\rho_L q^2 + P = P_{y=0}, \tag{4}$$

since at the nose on the plane of symmetry there is a stagnation point; the suffix L refers to the liquid properties. As the pressure P is continuous across the gas–liquid interface P in (4) is given by expression (1), in which $\Phi = \theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Also, by using Bernoulli's equation along the plane of symmetry it can be seen that

$$\tfrac{1}{2}\rho_L U_L^2 = \rho_G U_G^2. \tag{5}$$

Along the surface BC (figure 1), the speed is given by

$$q = U_L, \tag{6}$$

since this surface is assumed to be at a constant pressure.

The boundary conditions (4) and (6) indicate that (2) could be most conveniently solved in the velocity (q, θ) plane. So, making use of the hodograph transformation, the problem is illustrated in figure 2, where it is necessary to find a stream function ψ_L satisfying

$$\frac{\partial^2 \psi_L}{\partial q^2} + \frac{1}{q}\frac{\partial \psi_L}{\partial q} + \frac{1}{q^2}\frac{\partial^2 \psi_L}{\partial \theta^2} = 0.$$

The boundary conditions to be satisfied are $\psi_L = 0$ on DOA of figure 1, that is, using (1), (4) and (5),

$$\psi_L = 0 \begin{cases} \text{on} \quad 0 \leqslant q \leqslant U_L, \quad \theta = \pi, & (7a) \\ \text{on} \quad q = U_L \cos\theta, \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi. & (7b) \end{cases}$$

Also, on the surface BC, if $2h$ is the thickness of the jet as $x \to \infty$,

$$\psi_L = U_L h \quad \text{on} \quad q = U_L, \quad 0 < \theta < \pi. \tag{8}$$

The problem can now be solved by looking for a complex potential w to represent the flow in the region shown in figure 2, where $U_L h$ units of fluid are introduced at the point F and removed at G. From (7b) it can be seen that the curve OG is part of a circle with centre at $q = \frac{1}{2}U_L$, $\theta = 0$.

Bearing this in mind it is now possible to map this region into a half-plane through the following transformations. First, inverting with respect to the point G transforms

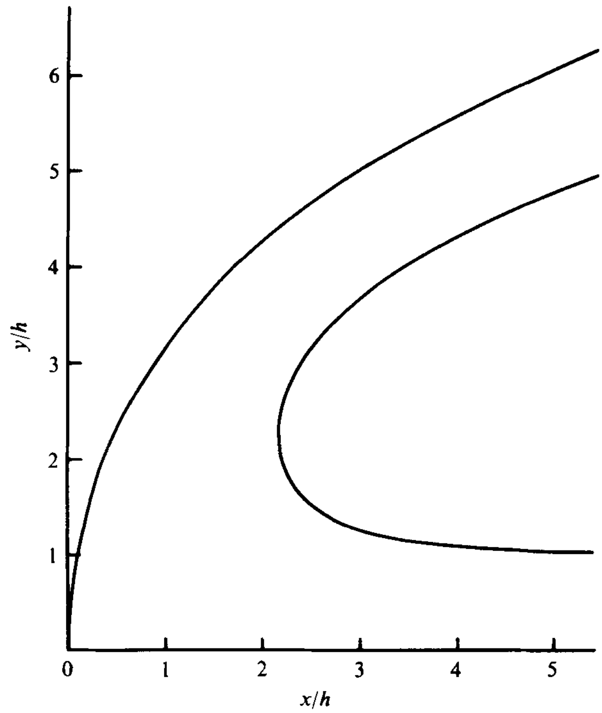

FIGURE 3. The shape of a jet moving into a gas.

the region of interest into a semi-infinite strip. This, after rotation, a shift of origin and suitable scaling, can be 'opened out' using a standard transformation so that the original region is mapped onto the upper half of, say, the ζ plane. The complex potential in the ζ plane is then easily written down and, using the transformations outlined above, is given in terms of q and θ by the expression

$$w = 2U_L h \ln\{1+\cosh[2\pi i Q/(Q-1)]\}, \tag{9}$$

where

$$Q = qe^{i\theta}. \tag{10}$$

In order to transform back to the physical co-ordinates x and y, use is made of the relation

$$dz = (e^{i\theta}/q)\,dw. \tag{11}$$

Referring to figure 1, to obtain the jet surface shape corresponding to OA, (11) is used together with (7b). Hence a parametric representation of the curve OA is given by

$$\left.\begin{aligned} \pi x &= h\ln[\tfrac{1}{2}(1+\cosh\alpha)], \\ \pi y &= h\int_0^\alpha \frac{2\pi}{\xi}\frac{\sinh\xi}{1+\cosh\xi}\,d\xi, \end{aligned}\right\} \quad 0 \leqslant \alpha < \infty. \tag{12}$$

Similarly the curve BC can be found by substituting $q = U_L$ in (11) to obtain

$$\pi(x+iy) = -h\int^\alpha \frac{\sinh\xi}{1-\cosh\xi}\frac{1}{\xi^2+\pi^2}[\xi^2-\pi^2+2\pi i\xi]\,d\xi. \tag{13}$$

The integrals can be evaluated and the curve BC located in the x, y plane by integrating (11) from $q = 0$ to $q = U_L$ along the line $\theta = \frac{1}{2}\pi$. It is useful to find the shape of the free surface given by (12) near the nose and also determine the asymptotic structure of the jet. For small values of α, (12) gives

$$y^2 = 4\pi h x$$

for the shape of the jet, and hence the radius of curvature of the jet at the nose is $2\pi h$. For large values of x the shape of the leading surface of the jet is given by

$$y = 2h \ln(\pi x/h).$$

The shape of the jet given by (12) is shown in figure 3.

3. The axially symmetric problem

Although the method developed is not suitable for dealing with the axially symmetric case it is of interest to inquire whether the two-dimensional model might throw some light on the axially symmetric problem with regard to certain overall characteristics. While it must be accepted that the shape of the axially symmetric jet cannot be determined with any confidence by reference to the two-dimensional theory, it might be possible to give an estimate for the radius of curvature at the nose of such a jet. This would of course be of some value as it would give information about the region of most importance, from the practical point of view, of the axially symmetric jet. In order to test the viability of some overall force balance approach it would be of value to compare the result from such a model with the result already obtained for the two-dimensional case.

Considering the equilibrium of the fluid making up the jet by equating the pressure forces to the rate of change of momentum of the fluid gives rise to the equation

$$\rho_G U_G^2 \int_{\Phi=\frac{1}{2}\pi}^{\phi=0} P \sin\Phi \, ds = 2\rho_L U_L^2 h, \tag{14}$$

where the pressure P is given by (1). Clearly the left-hand side of this equation depends on the shape of the free surface of the jet. However, since it seems reasonable to suppose that the main contribution to this integral comes from the region near the nose, the element of length ds may be taken to be that associated with a circular cylinder of radius R_0, say. The integral on the left-hand side of (14) can then be evaluated and, using (5), R_0 can be found in terms of h, i.e.

$$R_0 = 6h.$$

This should be compared with the result obtained from the two-dimensional analysis for the radius of curvature of the jet at the nose, namely $2\pi h$. This remarkably good agreement shows that the approach may be of value in the axially symmetric case and furthermore gives some indication of the confidence with which the result may be viewed.

If the radius of the undisturbed liquid jet is denoted by a, and R_0 again denotes the radius of curvature of, in this case, the spherical cap, the equation corresponding to (14) is

$$\rho_G U_G^2 R_0^2 \int_{\frac{1}{2}\pi}^{0} 2\pi(1-\cos^2\Phi)\cos\Phi\sin\Phi \, d\Phi = 2\pi a^2 \rho_L U_L^2,$$

which yields the result

$$R_0 = 2^{\frac{3}{2}} a.$$

4. The penetration problem

With reference to the penetration problem mentioned earlier, the liquid jet in this case moves into a region of plastic flow associated with the solid. Now, it is well established (see, for example, Rae 1970) that a solid, when subjected to pressures which greatly exceed the yield strength of the material, may be treated as an inviscid compressible fluid and indeed, if the pressures are sufficiently high, the equation of state approximates that of a perfect gas with a ratio of specific heats of 1·5. Furthermore, the liquid in the jet is ultimately deflected from its original direction through an angle of π radians.

This means that the pressure distribution on the leading surface should be of the general form indicated by (1) and the analysis carries through. The fact that in the analysis there is a mixture of incompressible and hypersonic compressible flow does mean that the Mach numbers are respectively much less than and much greater than unity in the two regimes. In other words, the material properties of the jet and the solid are quite different.

As the pressure drops from its very high stagnation value, the simple-gas approximation for the solid must fail and the effect of material strength becomes important. In practice this is shown by the solidification of the target material, and possibly the jet material in a solid–solid interaction, to form a lip near the rim of the impact crater. The experimental work of Christman & Gehring (1966) shows this quite clearly.

For cases when the Mach number is not very high, the method should still be useful since the pressure must drop from some stagnation value at the nose to a value which is effectively zero when $\theta = 0$, in which case (1) can be regarded as a first approximation and the resulting solution can then form the basis of some iterative procedure. Hence the initial estimate for the shape of the jet is the same as the shape for the hypersonic flow case shown in figure 3. Furthermore, the extension of the two-dimensional method to the axially symmetric case should carry over.

5. Comparisons with 'exact' solutions and experiment

An examination of the results of the calculations of Harlow & Pracht for the two-dimensional penetration jet allows one to estimate roughly the value of the radius of curvature of the jet at the nose. Concentrating on the result when the jet has penetrated the solid to a depth of about 3·5 undisturbed liquid jet widths, a rough measurement on the figures presented gives a radius of curvature at the nose of about $5h$. This should be compared with the value $2\pi h$ obtained for R_0 from the present theory. However, this quantity was extremely difficult to measure accurately so a comparison was also made of the shortest distance from the line $x = 0$ to the free surface BC (see figure 1). From the numerical results this distance made non-dimensional by the undisturbed jet width quickly assumed a steady value of about unity compared with 1·07 from the present analysis. This is very reasonable agreement under the circumstances and supports the assumption of steady flow.

Comparison of the theoretical results with experiments presents certain difficulties. For the case of a high-speed liquid jet travelling through gas it is observed that as the jet issues from the nozzle it appears to be broken up into a fine spray which prevents any clear picture of the flow being formed. It was for this reason that Leach & Walker decided that it would be more instructive to measure the pressure distribution on a flat plate perpendicular to the direction of the jet. The results indicated that the pressure was effectively zero at a distance of three jet radii from the axis. The present axially symmetric analysis based on a hemispherical leading surface gives a value of $2^{\frac{3}{2}}$ for this factor.

The early experiments of Pack & Evans, which dealt with the penetration into various metals of liquid-metal 'Munroe' jets, do form a basis for a careful comparison. In this case the penetration consisted of two or in some cases three distinct phases. The primary penetration of the jet was followed by secondary penetration as the hole continued to expand until the motion begun by the high pressure of the jet had been damped out by the resistance to the flow. Finally, behind the jet there was, in some cases, a slower moving and wider piece of metal called the plug which was capable of increasing the penetration.

The type of mainly lateral flow associated with the secondary penetration is, of course, highly dependent on the strength properties of the metal. Thus, in a metal such as lead with a relatively low yield strength this effect may well obscure the purely dynamic effects accounted for in the present analysis. This is borne out by the observations of Christman & Gehring. For this reason, it seems advisable to compare the theoretical results with those experimental results where secondary penetration was a small effect. Also, with tin forming the jet material no plug was present. With these restrictions, it was found that in steel targets the jet bored a hole of about four to five times its own radius. This result shows reasonable agreement with the factor $2^{\frac{3}{2}}$ obtained from the axially symmetric analysis.

Most of the experimental workers on the penetration of metals have used spherical solid projectiles, so that no useful comparisons can be made with the theory. Even the results of Tate (1969) for the impact of long hard metal rods on a relatively soft metal target are not compatible with the Mach number restrictions of the present theory. The experiments in this case were designed such that the strength of the material was a significant factor in the penetration process.

However, the extensive results of Christman & Gehring for the penetration of metal rods of various aspect ratios into large blocks of material do offer suitable comparisons. Amongst their results are some dealing with the penetration of steel rods with a length-to-diameter ratio of 10:1 into three different types of material. The X-ray pictures were taken during the penetrating process and show the radius of the crater near the nose to be about three times the projectile radius. These results are particularly important for comparison purposes because they give a clear picture of the primary phase of penetration, when, at the speeds used, the hydrodynamic analogy is valid and the flow is essentially steady in a frame of reference moving with the nose of the projectile. The agreement in this case with the present theory is remarkably good and again supports the assumption of steady flow.

As mentioned earlier, jets are used extensively as a cutting tool and in particular there is a substantial amount of experimental data on the penetration of various types of rock. This situation exhibits certain features which distinguish it from the penetration of metal. At relatively low speeds the jet penetrates the rock by a combination of fracture and erosion (see, for example, Rollins, Clark & Kalia 1973). However, at higher jet speeds, of the order of 5000 m/s, Farmer & Attewell (1965) suggested that the situation was more amenable to a hydrodynamic treatment and furthermore an approximately steady state would exist. Under these circumstances the present theory would be applicable.

Returning to the findings of Leach & Walker, they showed that the most commonly observed damage in various types of rock was a hole with a diameter about five times the jet diameter. The depth of the hole was affected by the interaction of the water escaping from the hole with the incident jet. Bearing in mind that some erosion of the hole would have taken place, these results were consistent with their pressure findings. However, it must be pointed out that the speed of 1000 m/s for the jet was not high enough for a hydrodynamic analogy to model adequately the penetration. Nevertheless, reasonable agreement is obtained owing to the pressure being adequately predicted.

6. Conclusions

The use of the method developed in this paper has been demonstrated for different types of physical problems where the main requirement is the determination of the shape of the free surface of a high-speed liquid jet. In all cases, the effect of viscosity has been ignored on the basis that the form of flow would be dominated by the dynamic pressures generated. Any compressibility effects in the liquid have also been ignored although at the speeds envisaged there would be a sudden expansion of the jet as it left the nozzle. This assumption also imposes some restrictions on the applicability of the model to the penetration problem. While the analogy between the plastic flow regime and a compressible gas is generally acceptable, the Mach number restrictions that this assumption imposes on the materials must be borne in mind.

The assumption that the flow is steady needs some qualification. For the case of a hypersonic liquid jet moving into a gas, this assumption limits the validity of the model to a region near the forward part of the jet, a few jet diameters downstream of the nozzle. However, this is the region of importance when such jets are used as cutting devices. Furthermore, there is some experimental evidence that a steady-state hydrodynamic analogy is the appropriate model for penetration problems.

For realistic equations of state for the solid during its deformation, possibly to include strength effects in the form of a stiffened gas, the solution for the penetration problem can be regarded only as a first step in some iterative procedure, but here again it seems reasonable to regard the flow as steady in the forward part of the jet after penetration to a distance of a few jet diameters. The comparison with the 'exact' calculations and experiment do vindicate the approximations used in this method.

The model presented in this paper illustrates the dominant effect of the inertia terms in the formation of the jet boundary and provides a particularly simple analytic approach to a type of problem hitherto analysed only in terms of large-scale numerical treatments. It is fair comment that the hydrocode methods do themselves suffer from some serious limitations in that the results are difficult to generalize, functional dependences are not apparent and consequently there is a lack of physical insight unless very many calculations over a parameter field are obtained. A critical review of these methods has been given by Dienes & Walsh (1970).

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